# COUNTING ORDERED COMBINATIONS

In counting combinations, sometimes the order matters. There is a difference between putting on your sock and then your shoe, versus putting on your shoe and then your sock! If we have two variables $x$ and $y$, there are four unordered combinations of degree three:

$$x^3, \quad x^2y, \quad xy^2, \quad y^3.$$

But if order matters, there are eight:

$$xxx, \quad xxy, \quad xyx, \quad yxx, \quad xyy, \quad yxy, \quad yyx, \quad yyy.$$

How can we count these ordered combinations? Here's an approach based on the "symbolic series" method of the text. Let $S$ be the sum of *all* the ordered monomials in two variables $x$ and $y$ (and we throw in 1 for the empty monomial):

$$S = 1+x+y+xx+xy+yx+yy+xxx+xxy+xyx+yxx+xyy+yxy+yyx+yyy+\cdots$$

Now since the order matters, every monomial (other than 1) must start with either $x$ or $y$. If we group these two sets of terms, we have

$$S = 1 + x(1 + x + y + xx + xy + yx + yy + \cdots)+ \\ y(1 + x + y + xx + xy + yx + yy + \cdots),$$

or $S = 1 + xS + yS$. We solve this formally to get

$$S = \frac{1}{1 - x - y}, \tag{1}$$

whatever that means. Let's try to make sense out of equation (1) by replacing each monomial $m$ by $t^{|m|}$, where $|m|$ is the degree of $m$. Then $S$ becomes the generating function $S(t)$ that counts ordered monomials by degree, and equation (1) becomes

$$S(t) = \frac{1}{1 - t - t} = \frac{1}{1 - 2t} = 1 + 2t + 4t^2 + 8t^3 + \cdots$$

Clearly the coefficient of $t^n$ is $2^n$, so there are $2^n$ ordered monomials of degree $n$ in $x$ and $y$ (or to put it another way, there are $2^n$ monomials of degree $n$ in noncommuting variables $x$ and $y$). This is really pretty obvious: in a monomial of degree $n$, you have $n$ factors and two choices ($x$ or $y$) for each factor. (Why doesn't this reasoning work if $x$ and $y$ are allowed to commute?)

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*Exercise 1.* Recall from the previous set of notes that there are $\binom{n+2}{2}$ monomials of degree $n$ in three commuting variables $x$, $y$, and $z$. How many distinct monomials of degree $n$ are there if $x$, $y$, and $z$ don't commute?

As the example of monomials shows, it's actually easier to count things when you keep track of the order. Here's another example. The ordered version of a partition is called a *composition*. Thus, $3+1$ and $1+3$ are considered distinct compositions of 4 (even though they represent the same partition). If we leave out the plus signs, we can write the eight compositions of 4 as

$$(1111), \quad (211), \quad (121), \quad (112), \quad (22), \quad (31), \quad (13), \quad (4).$$

Let $C$ be the symbolic sum of all compositions (with $(\ )$ as the empty composition of 0). Then

$$\begin{aligned} C = (\ ) + (1) + (11) + (2) + (111) + (21) + (12) + (3) + (1111) + (211) + (121) + (112) \\ + (22) + (31) + (13) + (4) + \cdots \end{aligned}$$

Now every composition must start with 1, or 2, or 3, etc. So if we define "multiplication" of compositions by juxtaposition (e.g., $(2) * (11) = (211)$), then

$$C = (\ ) + (1) * C + (2) * C + (3) * C + \cdots$$

or formally

$$C = \frac{(\ )}{(\ ) - (1) - (2) - (3) - \cdots}. \tag{2}$$

Let's try to make sense of equation (2) as we did with equation (1): replace each composition $c$ by $t^{|c|}$, where $|c|$ is the weight of $c$ (the sum of its parts). This takes $C$ to the generating function $C(t)$ that counts compositions by weight, so equation (2) becomes

$$C(t) = \frac{1}{1 - t - t^2 - t^3 - \cdots} = \frac{1}{1 - \frac{t}{1-t}} = \frac{1-t}{1-2t}.$$

Now

$$\begin{aligned} \frac{1-t}{1-2t} &= \frac{1}{1-2t} - \frac{t}{1-2t} \\ &= \sum_{n \geq 0} 2^n t^n - \sum_{n \geq 0} 2^n t^{n+1} \\ &= 1 + \sum_{n \geq 1} (2^n - 2^{n-1}) t^n \\ &= 1 + \sum_{n \geq 1} 2^{n-1} t^n, \end{aligned}$$

So there are $2^{n-1}$ compositions of $n$.

Actually there is an easier way to see that $n$ has $2^{n-1}$ compositions: think of a row of $n$ dots. If you insert dividers into some of the $n-1$ positions between the dots, you specify a composition of $n$. Since each of the $n-1$ positions can have a divider or not, that's $2^{n-1}$ choices. But the generating-function method is flexible enough to handle many related questions, such as the one in the next exercise.

*Exercise 2.* Let $Q_n$ be the number of compositions of $n$ in which all the parts are 1's and 2's. For example, $Q_5 = 8$ because there are eight such compositions of 5:

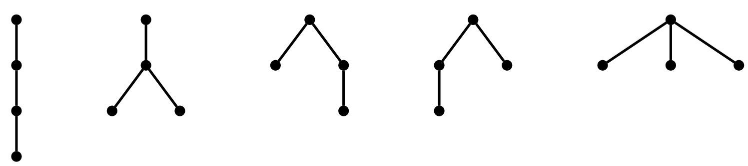
$$(11111),\quad (2111),\quad (1211),\quad (1121),\quad (1112),\quad (221),\quad (212),\quad (122).$$

Find the generating function

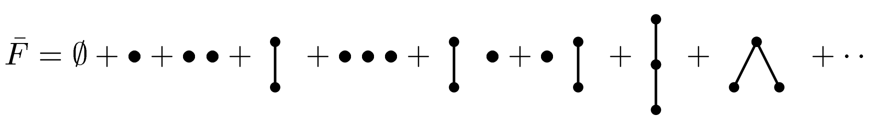
$$Q(t) = 1 + \sum_{n\geq 1} Q_n t^n.$$

Does this look like a generating function we've seen before?

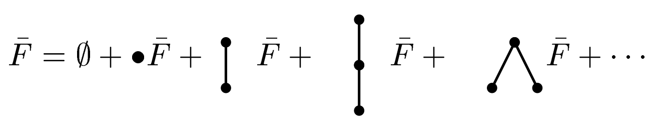
We can apply the same techniques to counting *planar* rooted trees. Let $P_n$ be the number of planar rooted trees with $n$ vertices. Then $P_4 = 5$ since there are five planar rooted trees with 4 vertices:



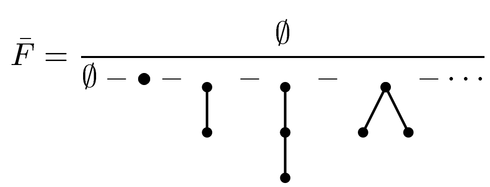
Now let's let $\bar{F}_n$ be the number of *ordered* rooted forests of planar rooted trees. If we form a symbolic sum of all ordered forests, it looks like



Since every nonempty ordered forest has a first tree, we can write this as



or

(3) 

As with equations (1) and (2), we interpret equation (3) by replacing each symbol with $t^w$, where $w$ is the symbol's weight (in this case, the number of vertices). Then equation (3) becomes

$$1 + \bar{F}_1 t + \bar{F}_2 t^2 + \cdots = \frac{1}{1 - P_1 t - P_2 t^2 - P_3 t^3 - \cdots}.$$

But every ordered forest of planar rooted trees with $n$ vertices corresponds to a planar rooted tree with $n+1$ vertices, so $\bar{F}_n = P_{n+1}$ and the preceding equation is

$$1 + P_2 t + P_3 t^2 + P_4 t^3 = \frac{1}{1 - P_1 t - P_2 t^2 - P_3 t^3 - \cdots}. \tag{4}$$

This is much easier than the equation we had in the previous set of notes. For if we let $P(t) = 1 + P_1 t + P_2 t^2 + \cdots$, then equation (4) multiplied by $t$ is

$$P(t) - 1 = \frac{t}{2 - P(t)}$$

or

$$P(t)^2 - 3P(t) + t + 2 = 0.$$

Solve this using the quadratic formula to get

$$P(t) = \frac{3 - \sqrt{9 - 4(t+2)}}{2} = \frac{3 - \sqrt{1-4t}}{2},$$

where we have chosen the negative square root to get $P(0) = 1$. But this says

$$P(t) = 1 + \frac{1 - \sqrt{1-4t}}{2},$$

and comparison with the generating function for Catalan numbers shows $P_n = C_{n-1}$ for $n \geq 1$. Of course this is the same result we had earlier via our isomorphism of planar trees with parenthesized products.